Study material for semester III
Department of Mathematics
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Topic: Real Analysis-I

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Contents

1 Introduction

This chapter defines the real numbers and studies their basic properties. The real numbers are characterized by three types of properties. The first type is algebraic properties (having to do with addition and multiplication). These properties are called the field axioms and are shared by other sets of numbers such as the rational numbers and the complex numbers, but not, for example, by the integers. The second consists of the order properties, which introduce an order structure (an inequality), and are also shared by the rational numbers, but not, for example, by the complex numbers. The third type is completeness that is not satisfied by the rational numbers and is the more difficult to state.

2 Properties of R

Axiom for addition: A set F is said to satisfy the axioms of addition if the function $+ : F \times F \to F$ satisfies the following properties:

(a) $x + y = y + x$ for all $x, y \in F$ (commutativity).

(b) $(x + y) + z = x + (y + z)$ for all $x, y, z \in F$ (associativity).

(c) There is an element of F denoted by 0 that satisfies $x + 0 = x$ for all $x \in F$ (identity).

(d) For every $x \in F$ there exists an element of F denoted $-x$, called its additive inverse, such that $x + (-x) = 0$ (Inverse).

Axiom for Multiplication: A set F is said to satisfy the axioms of multiplication if the function $. : F \times F \to F$ satisfies the following properties:

(a) $x.y = y.x$ for all $x, y \in F$ (commutativity).

(b) $(x.y).z = x.(y.z)$ for all $x, y, z \in F$ (associativity).

(c) There is an element of F denoted by 1 that satisfies $x.1 = x$ for all $x \in F$ (identity).

(d) For every $x \in F$ there exists an element of F denoted $\frac{1}{x}$, called its multiplicative inverse, such that $x.\frac{1}{x} = 1$ (Inverse).

Distributive Axiom: A set F with operations + and satisfies the distributive axiom if $x(y + z) = xy + xz$ for all $x, y, z \in F$.
Order axiom: A field F with operations + and is said to be an ordered field if there is a subset of F denoted by F^+ and called the positive set satisfying the following properties.

1. Closure of F^+ under + and i.e, $x + y$ and xy are in F^+ for all $x, y \in F^+$.

2. Trichotomy property: For every x in F, exactly one of the following is true:
   $x = 0, x \in F^+$ or, $-x \in F^+$.

2.1 Completeness Property of R

The completeness property, often referred to as order completeness, is a fundamental characteristic of the real numbers. Alongside the field and order axioms, the completeness property serves to fully define the real numbers.

Theorem 1. *There is no rational number x such that $x^2 = 2$.*

Proof. To prove the above result we use the method of contradiction that there is no positive rational number x such that $x^2 = 2$. Let us define the set $A = \left\{ q : x = \frac{p}{q}, \text{for some } p \in \mathbb{N} \right\}$.
If x is a rational number, then the set A is a nonempty subset of $\mathbb{N}$. By the well-ordering principle, it has a least element, which we denote by b. Then $x = \frac{p}{b}$ for some $p \in \mathbb{N}$.
This implies $2b^2 = p^2$.
Therefore it is clear that 2 is a factor of p^2 and since 2 is prime, 2 is a factor of p.
Then we can write $p = 2a$, for some integer a.
The above equation implies $b^2 = 2a^2$ and so $b = 2k$, for some $k \in \mathbb{N}$.
Hence $x = \frac{a}{k}$.
Thus $k \in A$ but $k < b$, contradicting that b is the least element of A. Therefore x is not a rational number. □

Definition 2.1 (Supremum)**.** *The supremum or least upper bound of S, denoted $\sup S$, is defined to be an element α of F such that*

(i) α is an upper bound for S;

(ii) if d is any other upper bound for S, then $d \geq \alpha$.

Similarly, the definition of the infimum or the greatest lower bound can be written as follows:

Definition 2.2 (Infimum)**.** *The infimum or greatest lower bound of S, denoted $\inf S$, is defined to be an element β of F such that*

(i) β is a lower bound for S;

(ii) if c is any other lower bound for S, then $c \leq \beta$.

2.2 Statement of completeness property:

An ordered field F is order complete, if every nonempty bounded subset of F has a supremum in F.
Archimedean property: For any real number x there exists a natural number n such that $n > x$.

3 Countable and uncountable set

There are infinitely many sets of real numbers. For the finite set it is clear that the number of elements of the sets are countable. But for infinite sets this logic is not work due to infinitely many points within the sets. Hence in this section we shall discuss which sets are countable and which are not.

Definition 3.1 (Countable set)**.** *A set A is countable if there is a bijection $f : \mathbb{N} \to A$.*

It is already shown that if a set A is finite then it is countable. For example the set of natural number $\mathbb{N}$ is countable.
From above definition we conclude that an infinite set is uncountable if it is not countable.

4 Open sets

Definition 4.1 (Open ball)**.** *The open ball centered at x with radius ϵ is defined by*

$$B(x, \epsilon) = \{y \in \mathbb{R} : |x - y| < \epsilon\}$$

Using the basic property of modulus of a function in real number the above expression can be written as $B(x, \epsilon) = (x - \epsilon, x + \epsilon)$.

Definition 4.2 (Interior point)**.** *A point x is called an interior point of a set $S \subseteq \mathbb{R}$ if there exists an open ball centered at x which wholly lies within the set S.*

The collection of all interior point of a set S is called interior of a set S and it is denoted by $Int(S)$.

Definition 4.3 (Open set)**.** *A set $S \subseteq \mathbb{R}$ is called open if for each $x \in S$ there exists an open ball centered at x which wholly lies within the set S.*

Therefore we conclude that if a set contains all of its interior points is open set. In general the empty set (Φ) and the whole set $(\mathbb{R})$ are open. Similarly, every open interval in $\mathbb{R}$ is open set. The following theorems are important:

Theorem 2.

- *The union of any arbitrary collection of open sets is open.*
- *The intersection of any finite collection of open sets is open.*

Proof. See any book for the proof of above result. □

Definition 4.4 (Closed set)**.** *A set $S \subseteq \mathbb{R}$ is closed if its complement S^c is open.*

For example, any closed interval in $\mathbb{R}$ is closed. By the similar argument the empty set (Φ) and the whole set $(\mathbb{R})$ are closed set.

Definition 4.5 (Accumulation or limit point)**.** *A point ξ is called a limit point of a set $S \subseteq \mathbb{R}$ if every neighborhood of ξ contains at least one point of the set S except ξ.*

Definition 4.6 (Isolated point)**.** *A point η is called an isolated point of a set $S \subseteq \mathbb{R}$ if it is not a limit point of S.*

Definition 4.7 (Derived set)**.** *The set of all limit point of a set $S \subseteq \mathbb{R}$ is called derived set of S. It is denoted by S'.*

Definition 4.8 (Closer or adherent point)**.** *A point ξ is called a closer point of a set $S \subseteq \mathbb{R}$ if every neighborhood of ξ contains at least one point of the set S.*

From above definition it is clear that a closer point of a set S either a member of the set or it is a limit point of that set.

Theorem 3 (Bolzano–Weierstrass Theorem)**.** *Every bounded subset of real number has at least one limit point or accumulation point.*

5 Sequence and series

Definition 5.1 (Sequence). *A sequence in a set S is a function from a certain kind of subset of integers into S.*

Definition 5.2 (Limit of a sequence). *A real number a is called limit of a sequence $\{a_n\} \in \mathbb{R}$ if for any $\epsilon > 0$ there exists a positive integer N such that $|a_n - a| < \epsilon$ for all $n > N$.*

In this case the sequence $\{a_n\}$ is called convergent and this limiting value is written as $\lim_{n\to\infty} a_n = a$.

Definition 5.3 (Monotone increasing). *A sequence $\{a_n\} \in \mathbb{R}$ is called monotone increasing if $a_{n+1} \geq a_n$ for all n.*

Definition 5.4 (Monotone decreasing). *A sequence $\{a_n\} \in \mathbb{R}$ is called monotone decreasing if $a_{n+1} \leq a_n$ for all n.*

A sequence $\{a_n\}$ is called monotone if it is either monotone increasing or decreasing.

Definition 5.5 (Subsequence). *A subsequence of a sequence $\{a_n\}$ is a sequence of the form $\{a_{n_k}\}$, where n_k is a strictly increasing sequence of elements of $\mathbb{N}$.*

Theorem 4 (Monotone convergence theorem). *Every bounded monotone sequence in $\mathbb{R}$ is convergent.*

Theorem 5 (Bolzano and Weierstrass). *Every bounded sequence of $\mathbb{R}$ has a convergent subsequence.*

Definition 5.6 (Cauchy sequence). *A sequence $\{a_n\}$ is called Cauchy sequence if for any $\epsilon > 0$ there exists a positive integer N such that $|a_m - a_n| < \epsilon$ for $m, n > N$.*

A common example of a Cauchy sequence is $\left\{\frac{1}{n}\right\}$.

Theorem 6. *A sequence $\{x_n\}$ is convergent iff it is Cauchy.*

Proof. See any standard book for the proof. □

Theorem 7 (Cauchy's convergence criterion). *A sequence $\{x_n\}$ is convergent iff for any $\epsilon > 0$ there exists a positive integer N such that $|x_m - x_n| < \epsilon$ for $m, n > N$.*

6 Exercise

1. Verify that the rational numbers $\mathbb{Q}$ satisfy the properties of an ordered field.
2. let $x \in \mathbb{R}$. Prove that if $x > 1$, then $x^n \geq x$ for all $n \in \mathbb{N}$.
3. Prove that $\sqrt{p} \notin \mathbb{Q}$ when p is prime.
4. Prove that the following property is equivalent to the Archimedean property: for each $x \in \mathbb{R}, x > 0$, there exists $n \in \mathbb{N}$ such that $\frac{1}{n} < x$.
5. Let A and B be sets. Prove that if B is countable and there exists a function $f : A \to B$ that is injective, then A is countable.

6. Prove that $\lim_{n\to\infty} |x_n| = 0$ if and only if $\lim_{n\to\infty} x_n = 0$.
7. Determine whether or not the following sequences are convergent and, if so, find their limit:
   (i) $\lim_{n\to\infty} (-1)^n \frac{5}{n}$
   (ii) $\lim_{n\to\infty} \frac{n^2 - 3n + 2}{n^2 + n}$
   (iii) $\lim_{n\to\infty} \frac{n^2 + 1}{n^3 + n + 1}$
   (iv) $\lim_{n\to\infty} (-1)^n n$
   (v) $\lim_{n\to\infty} \frac{1}{(-1)^n n + 3n}$
   (vi) $\lim_{n\to\infty} \sqrt[3]{n+1} - \sqrt[3]{n}$
8. Let $0 < a < 1$ and $x_n = a^{\frac{1}{n}}$ for $n \in \mathbb{N}$. Prove that $\lim_{n\to\infty} x_n$ exists, and find its value.
9. Newton's method: Let $a > 0$, and define a sequence $\{x_n\}$ by $x_{n+1} = \frac{1}{2}(x_n + \frac{a}{x_n})$, where x_1 is chosen to be any number greater than $\sqrt{a}$. Prove that $\lim_{n\to\infty} x_n = \sqrt{a}$.
10. Let $x_n = \sum_{i=1}^{n} \frac{1}{i(i+1)}$. Show that the given sequence is convergent and find its limit.
11. Find an example of a sequence $\{x_n\}$ such that $\lim_{n\to\infty} |x_{n+1} - x_n| = 0$ but $\{x_n\}$ is not Cauchy.